Combinatorics on Words

through the Word-Equations-lens

Florin Manea

Georg-August-Universität Göttingen

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Combinatorics on words is a fairly new field of mathematics, branching from combinatorics, which focuses on the study of words and formal languages. One looks at letters or symbols, and the sequences they form. Combinatorics on words affects various areas of mathematical study, including algebra and computer science. [..]
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The study of problems, algorithms, and data structures related to (efficient) string processing (Zvi Galil, 1984).

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THE References:
M. Lothaire - Combinatorics on Words (1983/1997), Algebraic Combinatorics on Words (2002), Applied Combinatorics on Words (2005).

CombWo via Word Equations

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- Word: $w \in A^*$; $w = w[1]w[2]\cdots w[n]$;
  Factor: $w[i:j] = w[i]w[i+1]\cdots w[j]$, Prefix: $w[1:i]$, Suffix: $w[i:n]$.

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- Inequation: $U \neq V$. Solutions $h : (X \cup A)^* \to A^*$ such that $h(U) \neq h(V)$.

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$$\mathtt{a\ b\ ab\ aba\ ab\ b\ b} \;=\; \mathtt{aba\ ba\ b\ aabb\ b}$$

Solution h maps $x \to \mathtt{b}$, $y \to \mathtt{aba}$

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  solutions of the form: $h(x) = a^i$
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  solutions of the form: $h(x) = w^i,\ h(y) = w^j$
- $\mathtt{a}\ x\ x = x\ \mathtt{b}\ y$
  no solutions!

The Satisfiability Problem

Decision Problem (Satisfiability of Word Equations):

Given a word equation $U = V$, does there exist a solution h satisfying $h(U) = h(V)$?

Size of input: $|U| + |V|$.
[Similar for systems (sets) of equations and inequations.]

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Observation [Karhumäki, Mignosi, Plandowski, 2000]

For every system of word equations and inequations one can construct an equivalent equation.

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Volker Diekert: More Than 1700 Years of Word Equations. CAI 2015: 22-28

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- Plandowski later showed that the satisfiabilty problem can be solved in PSPACE, and recently this was improved to linear space via recompression by Jez.
- On the other hand, it is fairly easy to show that the problem is NP-hard. Whether it is NP-complete remains a major open problem.

The Satisfiability Problem

Whether the satisfiabilty problem is NP-complete remains a major open problem.

One way to attack this problem (also for "fragments" of the theory of word equations).

Theorem (Plandowski, Rytter, ICALP 1998)

Suppose that for a given class of word equations, there exists a polynomial P such that any equation in the class which has a solution, has one whose length is at most $2^{P(n)}$ where n is the length of the equation. Then the satisfiability problem for that class is in NP.

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Main ideas: pure combinatorial

- Discover (and store) how the prefixes of p can be aligned with themselves: $\pi[i]$ = longest proper border of $p[1 : i]$
  (i.e., prefix of $p[1 : i]$ which is also a suffix of $p[1 : i]$, shorter than i)
- While reading the text t maintain (using the function π) the longest prefix of p which is a suffix of the prefix of t we've seen.

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Pattern Matching with Variables: Search

Given a pattern U and a word w, find all solutions of the equation $xUy = w$, where x, y are variables not occurring in U.

Motivation

- learning theory (inductive inference, PAC learning),
- language theory (pattern languages),
- pattern matching (parameterised matching, (generalised) function matching),
- matchtest for regular expressions with backreferences (text editors (grep, emacs), programming language (Perl, Java, Python)),
- string solvers (formal verification),
- database theory,
- bioinformatics.

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This is a long way from KMP's solution for $x\ p\ y = t$... So what is going on in between?

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Simple extension: Allow (multiple occurrences of) one variable in $p \to$ one-variable pattern U.

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Search *can be solved in* $O(rn)$ *time.*

There may be $\Theta(n^2)$ matches of U...
We can find a compact representation of all these matches.

More Variables: Regular Patterns

- **Regular Patterns**:
  $|U|_x = 1$, for all variables x occurring in U.
  E. g., $U = \mathrm{ab}x_1x_2\mathrm{b}x_3\mathrm{aaa}x_4\mathrm{b}$.

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Greedy strategy: the variables are just "spacers". So use KMP to search greedily for the terminal parts!
Search is a bit more involved: it is solvable in $\mathcal{O}(|U| + |w| + occ)$, where occ is the number of matches of U in w.

Combine the two extensions...
k-Repeated Variable Patterns

- **k-Repeated-Variable Patterns**:
  $|\{x \in \mathrm{var}(U) \mid |U|_x \geq 2\}| \leq k$.
  E. g., $U = x_1 \mathrm{ab} x_2 \mathrm{a} x_2 \mathrm{a} x_3 \mathrm{ba} x_2 \mathrm{bb} x_4 x_2 x_5$ is a 1-repeated-variable pattern.

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Lemma (Fernau, M., Mercaş, Schmid, STACS 2015)

Match *for* 1*-repeated-variable patterns is solvable in* $\mathcal{O}(|w|^2)$.

Theorem

Match *for* k*-repeated-variable patterns is solvable in* $\mathcal{O}\left(\frac{|w|^{2k}}{((k-1)!)^2}\right)$.

Match for k-repeated-variable patterns is $W[1]$-hard w.r.t. parameter k.

Combine the two extensions...
Non-Cross Patterns

- **Non-Cross Patterns**: the pattern has a "regular" structure, but instead of single variables, we have one-variable patterns.
  $U = \ldots x \ldots y \ldots x \ldots$ is not possible.
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Match *for non-cross patterns is solvable in* $\mathcal{O}(|w|m\log|w|)$, *where* m *is the number of one-variable blocks of the pattern.*

Same complexity for Search.

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Same complexity for Search.

Open problems:

More interesting (motivated), better parameters leading to poly-time matching? Faster algorithms? Fine grained complexity?

A General Theory

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Then G is a "valid U-graph".

Patterns with Bounded Treewidth

Example: $U = x_1 x_2 \mathtt{a} x_2 \mathtt{a} x_3 x_1 x_2 x_1$

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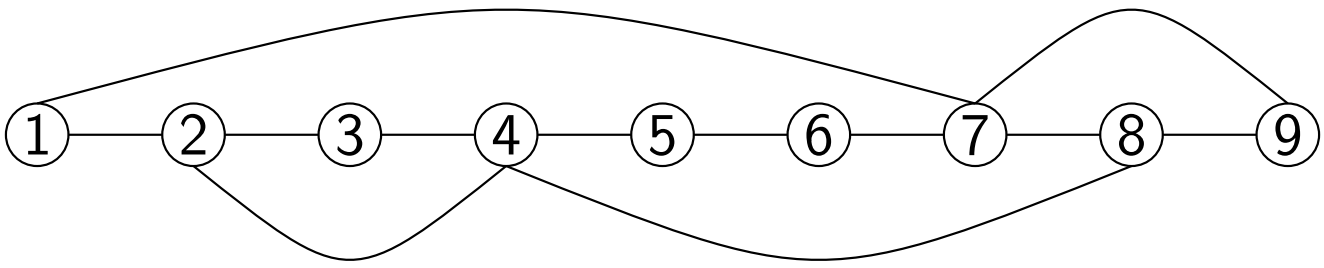

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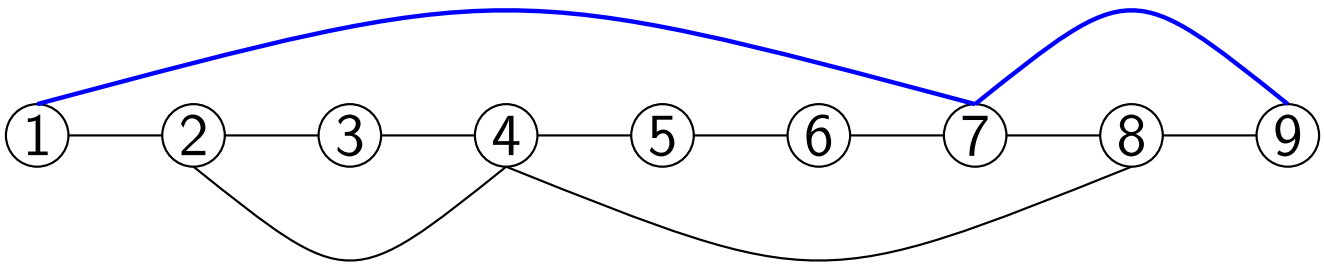

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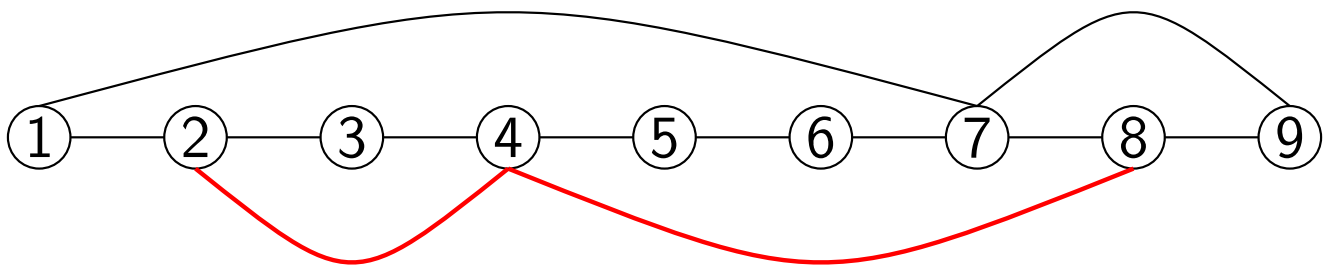

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Reidenbach & Schmid (Inf. Comput. 2014):

A class of patterns C has bounded treewidth if there exists $k \in \mathbb{N}_0$ and a polynomial time computable function mapping each pattern $U \in C$ to a valid U-graph G_U such that G_U has treewidth at most k.

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Can we match efficiently classes of patterns whose treewidth is unbounded?

Generalized Repetitions

- Let U and V be a patterns. We say that U is a generalized repetition of V if the skeleton of U (all terminals removed from U) is a repetition of the skeleton of V.

  Example: $\mathtt{a}\mathtt{a}x_1\mathtt{a}\mathtt{b}x_2 \cdot \mathtt{a}\mathtt{b}x_1\mathtt{b}\mathtt{a}\mathtt{b}x_2 \cdot \mathtt{a}\mathtt{a}x_1\mathtt{b}\mathtt{b}\mathtt{a}x_2$

Matching Generalized Repetitions of Regular Patterns

Theorem (Day, Fleischmann, M., Nowotka, Schmid, DLT 2018)

We can solve Match *for a generalized repetition of a regular pattern U with m variables and a word w of length n in $O(nm)$ time.*

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The class of generalized repetitions of regular patterns has unbounded treewidth.

Can we match efficiently **even more** classes of patterns whose treewidth is unbounded?
See:
Freydenberger, Peterfreund: Finite models and the theory of concatenation. Arxiv 2019.
A new approach to {word equations (the theory of concatenation), pattern matching, document spanners} based on finite model theory (and motivated by data bases!)

Open

Can we explore in a meaningful way the graph-connection? State-of-the-art approximation algorithm for cutwidth obtained this way [Casel et al, ICALP 2019].

Pattern Avoidability

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U pattern, x a new variable.

Is U avoidable?

Given a pattern U does there exist an infinite word ω such that the equation $xU = w$ is unsatisfiable for all prefixes w of ω.

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An example:

Thue (1906): Cubes are avoidable

The pattern $U = zzz$ is avoided by the infinite word $0110100110010110\ldots$, generated by applying iteratively the morphism $h(0) = 01$ and $h(1) = 10$ to 0.
That is $U = zzz$ is avoided by $\lim_{i\to\infty} h^i(0)$.

Squares xx, overlaps $yxyxy$, etc. are also avoidable.

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A general theory

Zimin words: $Z_1 = x_1$ and $Z_{n+1} = Z_n x_{n+1} Z_n$ for $n \geq 1$.
A pattern U over n distinct pattern variables is unavoidable if and only if the pattern U matches a factor of the n-th Zimin pattern Z_n.

Periodicity Enforcing Equations

Other “simple" word equations?

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Other “simple" word equations? Like some equations where it is clear which are the solution-sets.
The theorems of Lyndon and Schützenberger.

Theorem

- *The solutions of the equation* $x^i = y^j$, *with* $i, j > 0$, *are* $x = u^k$ *and* $y = u^\ell$ *for some* $u \in A^+$ *and* $k, \ell \geq 0$.
- *The solutions of the equation* $xy = yx$ *are* $x = u^k$ *and* $y = u^\ell$ *for some* $u \in A^+$ *and* $k, \ell \geq 0$.

Theorem

The solutions of the equation $x^i y^j = z^k$, *with* $i, j\ell \geq 2$, *are* $x = u^e$, $y = u^f$, $z = u^g$, *for some* $u \in A^*$ *and* $e, f, g \geq 0$.

Theorem

The solutions of the equation $xy = yz$ *are* $x = uv$, $z = vu$, $y = (uv)^e u$, *for* $u, v \in A^*$ *and* $e \geq 0$.

Periodicity Enforcing Equations

Theorem (folklore)

The system $\{xx = yxz, y \neq \epsilon, z \neq \epsilon\}$, *has only solutions* $x = u^e$, $y = u^f$, $z = u^g$, *for some* $u \in A^*$ *and* $e, f, g \geq 0$.

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Theorem (Saarela, STACS 2017)

The system $x_0^{k_i} = x_1^{k_i} \dots x_n^{k_i}$ *with* $i \in \{1, 2, 3\}$ *and* k_1, k_2, k_3 *distinct positive integers, has only solutions* $x_j = t^{\ell_j}$, *where* $t \in A^*$ *and* $\ell_j \geq 0$.

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Theorem (Saarela, ICALP 2020)

Deciding whether a given constant-free equation has a nonperiodic solution is NP-hard.

On Solving Word Equations with Simple Structure

Structural restrictions made sense for pattern matching.
Which are the structurally simplest word-equations, for each the satisfiability is still hard? (M., Nowotka, Schmid, DLT 2016)

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Quadratic equations (Diekert, Robson, 1999)

Is the satisfiability of quadratic word equations (a variable occurs at most 2 times) is in NP?

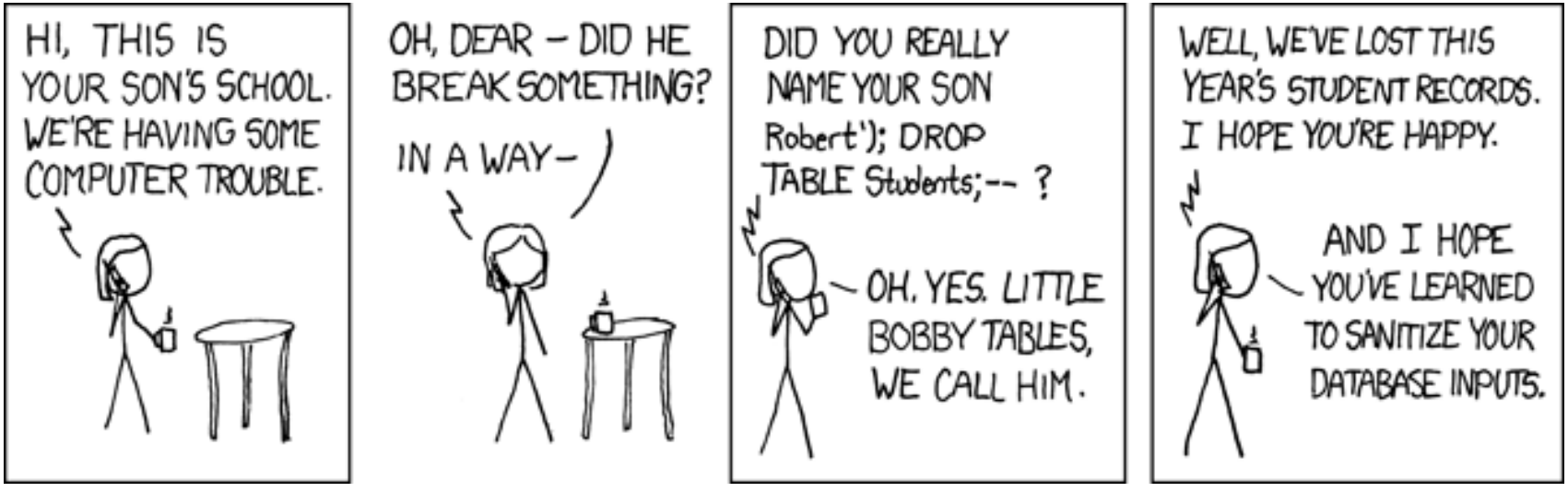
The Satisfiability Problem - Practice

More General Decision Problem(s):

Given a word equation $U = V$, does there exist a solution h satisfying $h(U) = h(V)$ and such that h satisfies some additional properties or constraints?

Motivation

- There are many reasons to care about solving word equations, with applications in various areas from algebra to database theory.
- However, even the linear space recompression algorithm is massively non-deterministic, and thus solving word equations in practice remains largely infeasible.
- Nevertheless, in recent years, there has been elevated interest in solving word equations from the point of view of verification and e-security.



Source: XKCD

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- Several SMT-solvers for strings are being developed which incorporate (incomplete) algorithms for solving word equations.
- Usually, however, it is desirable to be able to look for solutions satisfying some additional conditions, such as comparisons of the lengths of (images of) variables, restricting the (images of) variables to regular languages etc.

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- In fact, many simple extensions lead immediately to undecidability. Some lead to decidability.
  E.g., subword-constraints lead to undecidability (Halfon, Schnoebelen, Zetzsche, LICS 2017).
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  See (Day et al. RP 2018) for a longer discussion.
- One particularly interesting case is solving word equations modulo length constraints, whose decidability status has been open for many years.

Regular Constraints

Example:

$$x\ \mathtt{a}\ y\ x\ \mathtt{b} = y\ y\ \mathtt{ab}\ x$$

$$L_x = \mathtt{a}^+\mathtt{b}^+ \quad L_y = \mathtt{b}^+\mathtt{a}^+$$

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- Satisfiability of word equations with regular constraints is PSPACE-complete when the constraints are given by NFAs or DFAs.

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- Decidabillity of satisfiability for word equations with linear length constraints is an open problem dating back to 1968.
- Decidabillity of satisfiability for regular/quadratic word equations with linear length constraints is also open! (see, e.g., Lin, Majumdar, ATVA 2018)

Even More Word Equations

- Expressibility:
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- Independent Systems:
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- This week @ ICALP:
  Aleksi Saarela: Hardness Results for Constant-Free Pattern Languages and Word Equations
  Joel D. Day, M.: On the Structure of Solution Sets to Regular Word Equations.

The End

Thank you!