Logic meets **graph theory** and **algorithm design**

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University of Warsaw

Logic Mentoring Workshop @ CSL 2022

February 14th, 2022

Graph algorithms

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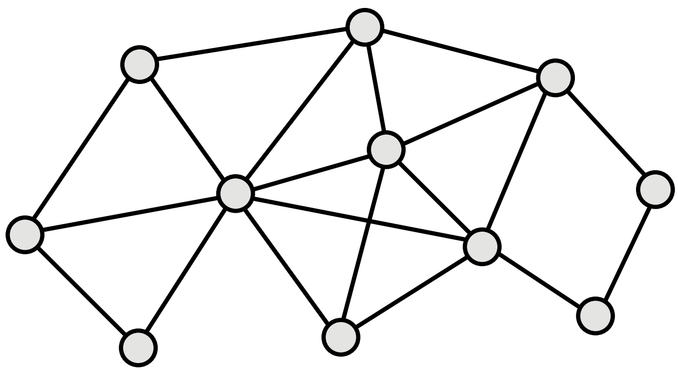

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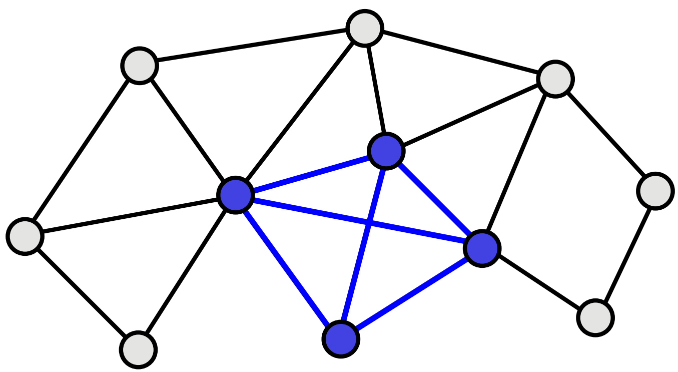
- CLIQUE: Does G have k pairwise adjacent vertices?



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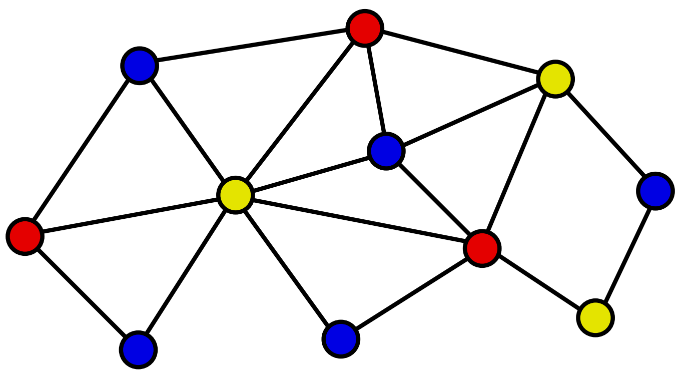

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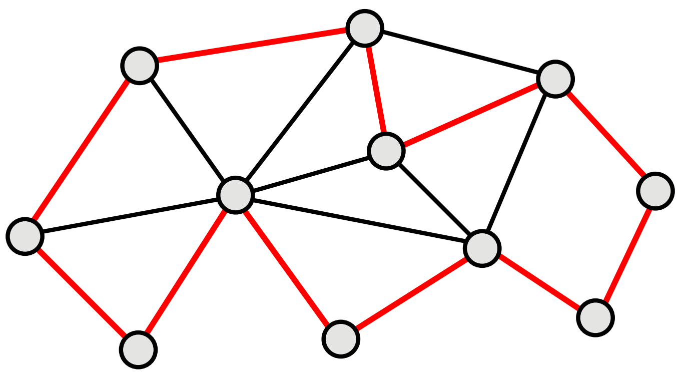
- Hamiltonicity: Does G have a cycle visiting every vertex once?



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Model-checking $\mathcal{L}$ on $\mathscr{C}$: Given $\varphi \in \mathcal{L}$ and $G \in \mathscr{C}$, decide $G \models \varphi$.

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Monadic Second-Order logic, second variant (MSO_2):

– vars for (sets of) vertices & edges, can check membership & incidence

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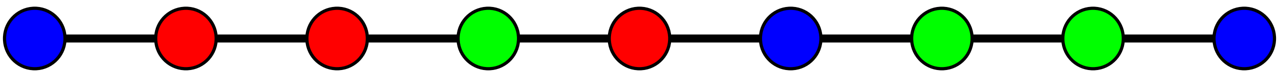
Obs: Every MSO_2-definable problem can be decided in time $\mathcal{O}(n)$ on **colored paths**.



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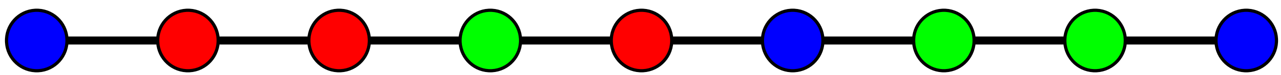

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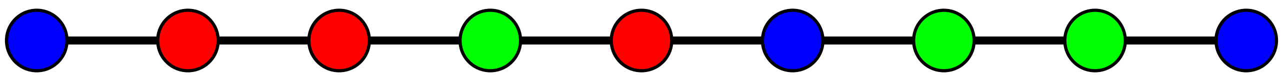

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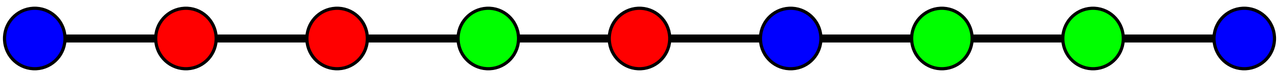

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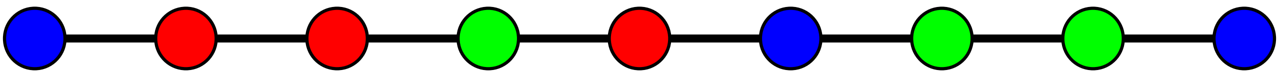

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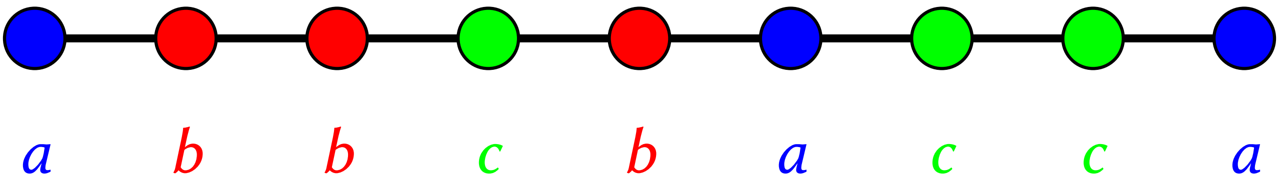

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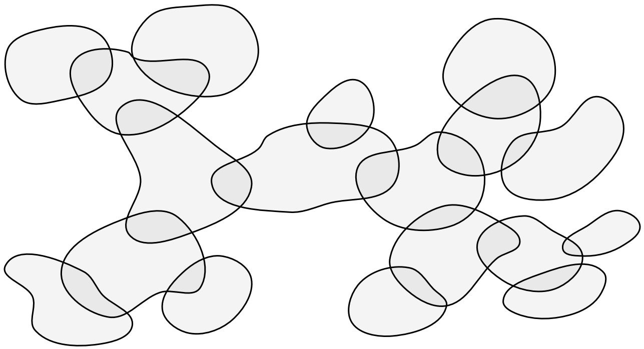
A graph has **treewidth** k if it can be confined to a tree of **bags**, each of size $\leqslant k$.



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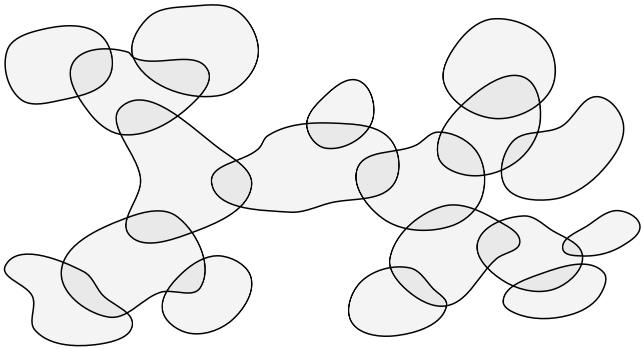

Theorem (Courcelle)

For every fixed k, every MSO_2-definable problem can be decided in linear time on graphs of treewidth $\leqslant k$.

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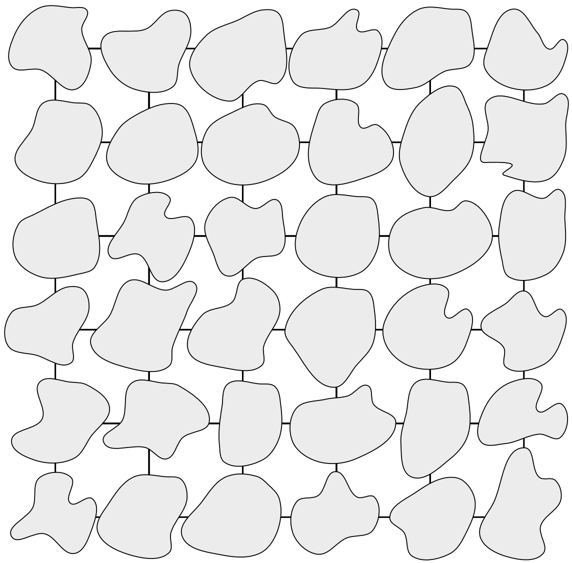
Theorem (Excluded Grid Minor)

There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that if the treewidth of G is larger than $f(k)$, then G contains a $k \times k$ **grid minor**.



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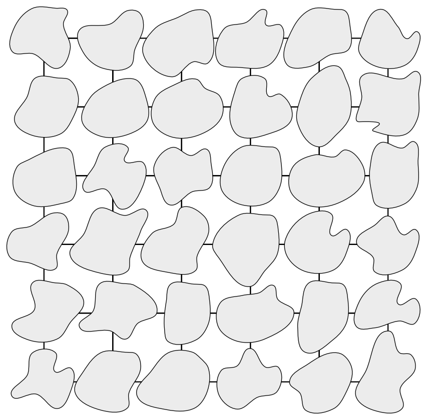

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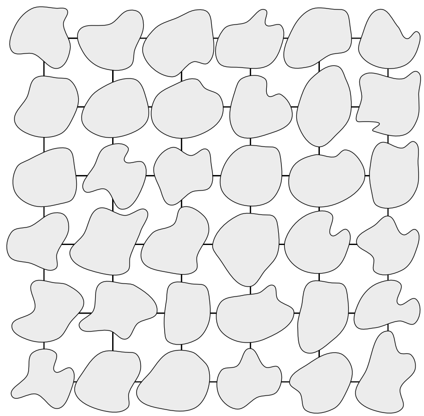

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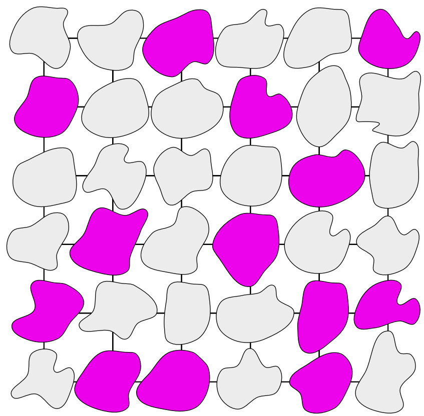

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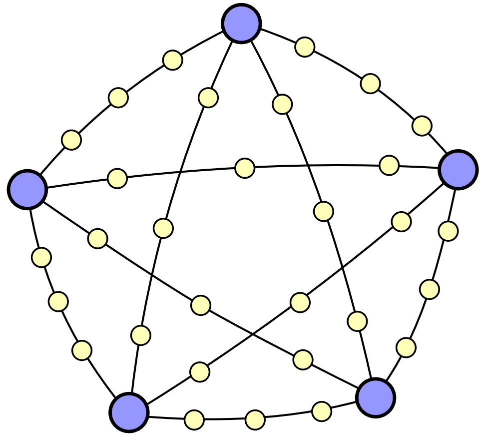
A class of graphs $\mathscr{C}$ is **nowhere dense** if for every $d \in \mathbb{N}$ there is $t(d)$ such that graphs from $\mathscr{C}$ **exclude** the d-**subdivision** of the **clique** $K_{t(d)}$.



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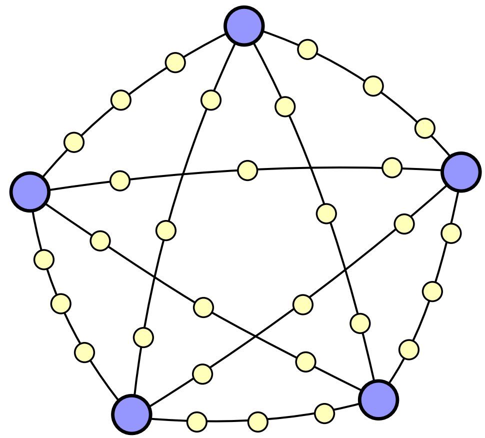

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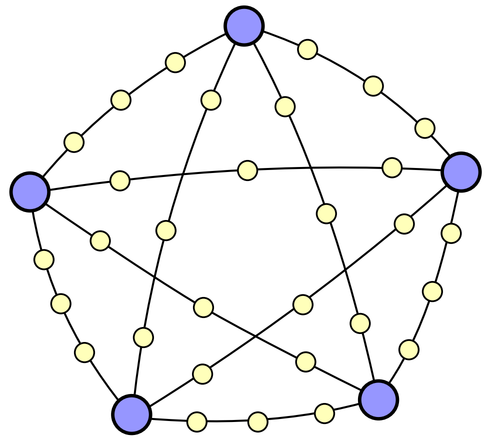

Theorem (Grohe, Kreutzer, Siebertz)

Suppose $\mathscr{C}$ is a class of graphs closed under taking subgraphs. Then:

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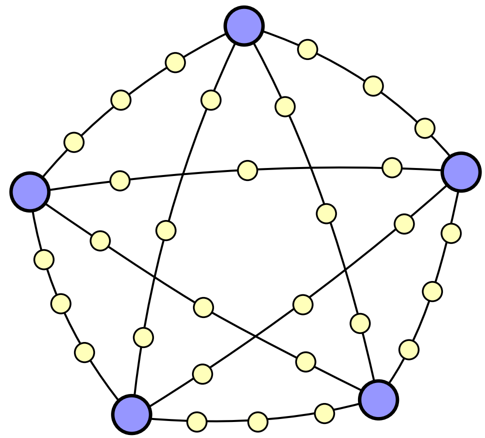

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Q: Proposition for ?? in the following:

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- any class of bounded cliquewidth.

Q: Proposition for ?? in the following:

$$\mathrm{FO} \quad \longleftrightarrow \quad ??$$

Definition (Monadic dependence)

A class of graphs $\mathscr{C}$ is **monadically dependent** if one cannot FO-interpret all graphs in colored graphs from $\mathscr{C}$.

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So far...

Intermediate logic

$$\begin{array}{ccc} \mathrm{MSO}_2 & \longleftrightarrow & \text{bnd treewidth} \\ \cup & & \cap \\ \mathrm{MSO}_1 & \longleftrightarrow & \text{bnd cliquewidth} \\ \cup & & \\ \mathrm{FO} & \longleftrightarrow & \text{nowhere dense} \end{array}$$

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Can we find natural variants of logic between FO and MSO?

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\mathrm{FO} + \mathrm{conn} & \longleftrightarrow & \text{top-minor-free} & \\
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Proposition: logic $\mathrm{FO} + \mathrm{conn}$ corresponds to **topological-minor-free classes**.

FO + conn

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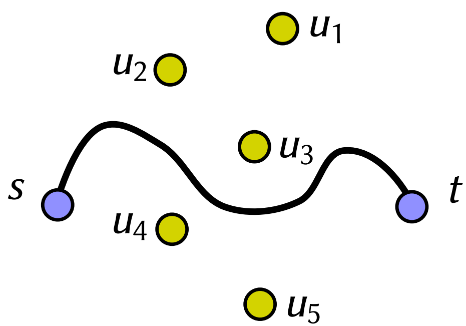
Def: $G \models \mathrm{conn}(s, t, u_1, \ldots, u_k)$

if s and t can be **connected** by a **path** avoiding $u_1, \ldots, u_k$.



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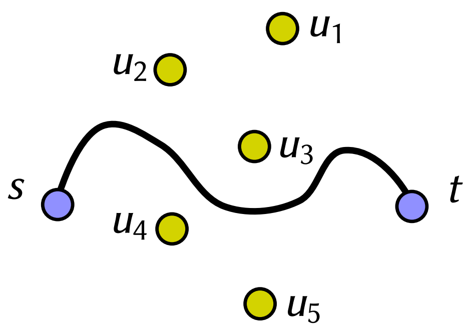

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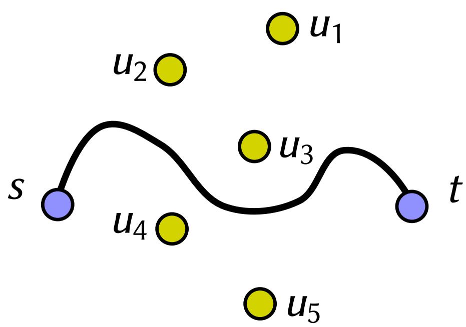

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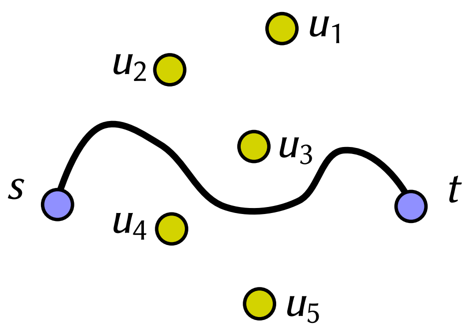
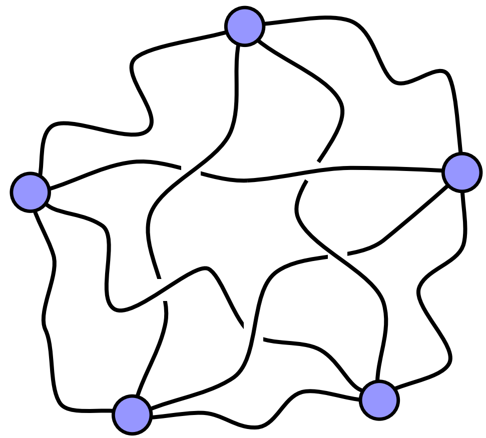
no graph $G \in \mathscr{C}$ contains a subdivision of K_t.





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Suppose $\mathscr{C}$ is a class of graphs closed under taking subgraphs. Then:

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Key: A known decomposition into parts where FO + conn reduces to FO.

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Thanks for attention!